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COMMENSURABILITY OF LINK COMPLEMENTS

Dedicated to Prof. Taizo Kanenobu, Makoto Sakuma, Yasutaka Nakanishi on their 60-th birthday

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Abstract

In 2013, Chesebro and DeBlois constructed a certain family of hyperbolic links whose complements have the same volume, trace field, Bloch invariant, and cusp parameters up to $PGL(2,\mathbb{Q})$. In this paper, we show that these link complements are incommensurable to each other. We use horoball packing to prove this.

1. Introduction

It was shown that there exist an arbitrary number of hyperbolic 3-manifolds with the same invariant trace field and volume but are mutually incommensurable which are distinguished by their cusp parameters ([1] Theorem 3.(2), [9]). In [1] Theorem 3.(3), E. Chesebro and J. DeBlois have constructed a certain family of links whose complements have the same volume, trace field, Bloch invariant and cusp parameter up to $PGL(2,\mathbb{Q})$. For these link complements, they say "We do not know if these are commensurable, although we suspect they are not".

In this paper, we show the following theorem.

Theorem 1. *The hyperbolic link complements in* [1] Theorem 3.(3) *are incommensurable to each other.*

To prove the incommensurability of these hyperbolic link complements, we investigate the horoball packings of them.

2. Preliminary

In this section, we construct hyperbolic links in [1] Theorem 3.(3). For an arbitrary number $n \in \mathbb{N}$, consider the link L as in Figure 1. The dotted lines there indicate the presence of 2-spheres that each of which meets L in 4 points. The 2-spheres separate the link L into tangles from left-to-right into a tangle S and n copies of T and the mirror image $\bar{S}$ of S. We denote the spheres by $S^{(m)}$ ($m = 0, \cdots, n$) so that $S^{(0)}$ bounds S, $S^{(n)}$ bounds $\bar{S}$, and $S^{(m-1)}$ bounds a copy of T with $S^{(m)}$.

Let L_m ($m = 1, \ldots, n$) be the link obtained from L by mutation along $S^{(m-1)}$ and $S^{(m)}$ as in Figure 2.

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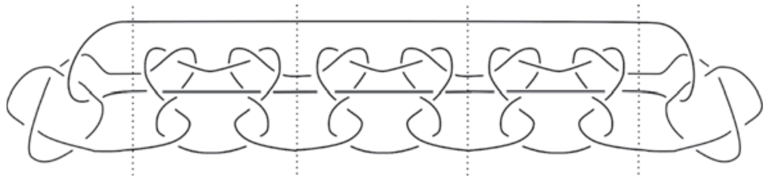

Fig. 1. The link $L(n = 3)$.

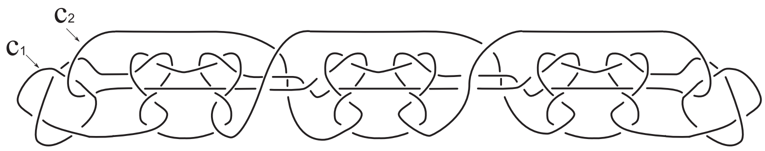

Fig. 2. The link L_m ($n = 3$ and $m = 2$).

3. Commensurability invariant

Two hyperbolic 3-manifolds are commensurable if they have a common cover, of finite degree. Several commensurability invariants are known. Two commensurable manifolds have necessarily commensurable volumes. For a Kleinian group Γ, put $\Gamma^{(2)} = \langle \gamma^2 | \gamma \in \Gamma \rangle$. The invariant trace field $\mathbb{Q}(\mathrm{tr}\, \Gamma^{(2)})$ is a commensurability invariant of $M = \mathbb{H}^3/\Gamma$. In particular, if M is a hyperbolic link complement, the invariant trace field coincides with trace field $\mathbb{Q}(\mathrm{tr}\, \Gamma)$ [4]. Suppose that M has a degree one ideal triangulation by ideal simplices $\Delta_1, \cdots, \Delta_n$. Then the Bloch invariant $\beta(M) = \Sigma_{i=1}^{n} [z_i]$ is an element in the pre-Bloch group $\mathcal{P}(\mathbb{C})$, where $z_i \in \mathbb{C}$ is the parameter of the ideal tetrahedron Δ_i. Commensurable hyperbolic manifolds have $\mathbb{Q}$-dependent Bloch invariants [8]. If two cusped hyperbolic manifolds are commensurable, they have the same set of cusp parameters up to $PGL(2, \mathbb{Q})$.

Let $c_1, \dots, c_k$ be the cusps of a cusped hyperbolic manifold M. Expand the horoball neighborhood of c_j until it collides itself or some other horoball neighborhoods ($j = 1, \cdots, k$). If M has only one cusp, this horoball neighborhood of c_1 is uniquely determined, which is called maximal cusp. In general, if $k \geq 2$, these horoball neighborhoods are not uniquely determined. However, if each horoball neighborhood collides itself, they are uniquely determined. These horoball neighborhoods lift to an infinite set of horoballs in $\mathbb{H}^3$ with disjoint interiors and some points of tangency on their boundaries. This set of horoballs is also uniquely determined up to $PSL(2, \mathbb{C})$, we denote it by $\mathcal{H}(M)$. The commensurability class of non-arithmetic orbifolds contains an element which is covered by any other manifold and orbifold in the class [3]. If two manifolds M_1 and M_2 cover a common orbifold Q, they admit choices of horoball neighbourhoods lifting to isometric horoball packings ([2] Lemma 2.3.). We can get the following proposition.

Proposition 1. *Suppose that non-arithmetic cusped hyperbolic manifolds M_1 and M_2 are commensurable. If each horoball neighborhood of the cusps of M_i ($i = 1, 2$) collides itself, then $\mathcal{H}(M_1) = \mathcal{H}(M_2)$ up to $PSL(2, \mathbb{C})$.*

The invariant trace field of $S^3 - L_m$ is $\mathbb{Q}(\sqrt{-1}, \sqrt{2})$ ([1] Proposition 4.1.). It is well known that any non-compact arithmetic manifold M has invariant trace field $k(M) = \mathbb{Q}(\sqrt{-d})$ for some $d \in \mathbb{N}$ [5]. Thus $S^3 - L_m$ is non-arithmetic.

In section 5, we show that $\mathcal{H}(S^3 - L_m) \neq \mathcal{H}(S^3 - L_{m'})$ $(m \neq m')$ up to $PSL(2, \mathbb{C})$.

4. Horoball neighborhoods of cusps of $B^3 - S$ and $S^2 \times I - T_0$

To see the horoball neighborhoods of the cusps of $S^3 - L_m$, we consider the tangle S in B^3 and the tangle T_0 in $S^2 \times I$ as shown in Figure 3. In the next section, we show that if we expand the horoball neighborhoods of the cusps of $S^3 - L_m$ until the meridian length of them are $\sqrt{3}$, each horoball neighborhood collides itself. In this section, we expand horoball neighborhoods of the cusps of $B^3 - S$ and $S^2 \times I - T_0$ until the meridian length of them are $\sqrt{3}$.

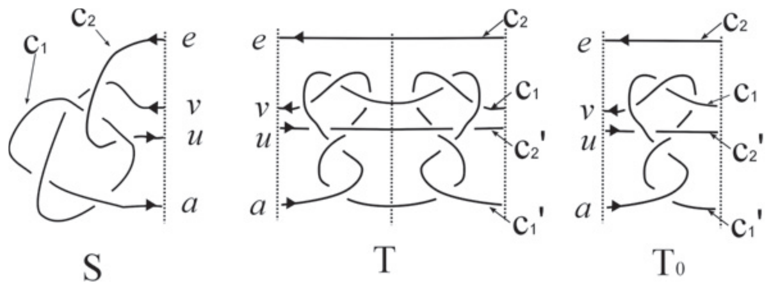

Fig. 3. Tangles S, T and T_0.

Let P_1 be the regular ideal octahedron and X_1, X_2, X_3 and X_4 the ideal triangles of P_1 as shown in Figure 4. The ideal triangle X_1 is identified to X_2 by $s = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and the ideal triangle X_3 is identified to X_4 by $t = \begin{pmatrix} 2i & 2-i \\ i & 1-i \end{pmatrix}$. Put c_1 and c_2 be the cusps of $B^3 - S$ as in Figure 3. In [1] section 2, the following proposition is proved.

Proposition 2. *There is a homeomorphism from $B^3 - S$ to $P_1/\{s^{\pm 1}, t^{\pm 1}\}$, which is a hyperbolic manifold with totally geodesic boundary. The ideal point* 0 *corresponds to the cusp c_1. Other ideal points correspond to the cusp c_2.*

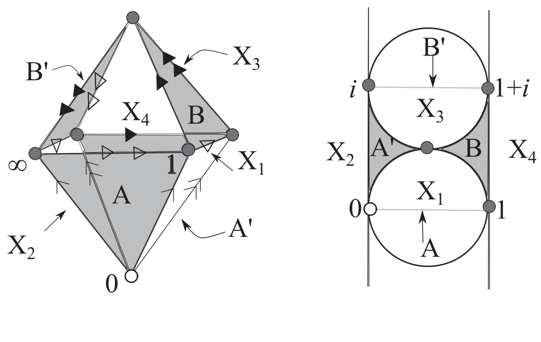

Fig. 4. Ideal polyhedral decomposition of $B^3 - S$.

For a horoball h, denote the Euclidean height of h by $D(h)$. If the center of h is not ∞, $D(h)$ is the Euclidean diameter of h. We will use the following well-known lemma which can be proved by direct calculations.

Lemma 1. *Let h be a horoball centered at ∞ with Euclidean height k. If $\gamma \in PSL(2,\mathbb{C})$ does not fix ∞, then $D(\gamma(h)) = 1/|c^2k|$, where c is the first entry in the second row of γ.*

Proposition 3. *Let $N(c_j)$ be the horoball neighborhood of c_j such that the meridian length of $\partial N(c_j)$ is $\sqrt{3}$ $(j = 1, 2)$. Then $\partial N(c_j)$ is as in* Figure 5 *and $N(c_j) \cap N(c_k) = \emptyset$ $(j, k = 1, 2)$.*

(Proof of Proposition 3.) Let h_0 be a horoball whose center is 0 and $D(h_0) = \sqrt{3}$. $\partial h_0 \cap P_1$ is a square with sides of length $\sqrt{3}$. We identify the sides of the square $\partial h_0 \cap P_1$ by s. The horospherical cross-section of the cusp c_1, which is $\partial N(c_1)$, is as in the left side of Figure 5.

Put $\mathcal{V}_1 = \{\infty,\ 1,\ i,\ 1,\ (1)/2\}$. Let $\{h_z | z \in \mathcal{V}_1\}$ be a collection of horoballs invariant under the action of the symmetry group of P_1, such that h_z is centered at z for each $z \in \mathcal{V}_1$ and h_∞ is at height $5/\sqrt{3}$. We identify the sides of these squares $\partial h_z \cap P_1$ by s and t. The horospherical cross-section of the cusp c_2, which is $\partial N(c_1)$, is as in the right side of Figure 5.

The ideal point ∞ is identified to 1 by $t^{-1} = \begin{pmatrix} 1-i & 2-i \\ -i & 2i \end{pmatrix}$. By Lemma 1, we have $D(h_1) = \sqrt{3}/5$. By the symmetry of P_1, $D(h_1) = D(h_i) = \sqrt{3}/5$. The ideal point ∞ is identified to $(1)/2$ by $t^{-2} = \begin{pmatrix} 1 & -3-i \\ 1-i & -3 \end{pmatrix}$. By Lemma 1, we have $D(h_{(1)/2}) = \sqrt{3}/10$.

It is not hard to see $h_z \cap h_{z'} = \emptyset$ for $z \neq z'$. Then $N(c_j) \cap N(c_k) = \emptyset$ $(j, k = 1, 2)$.

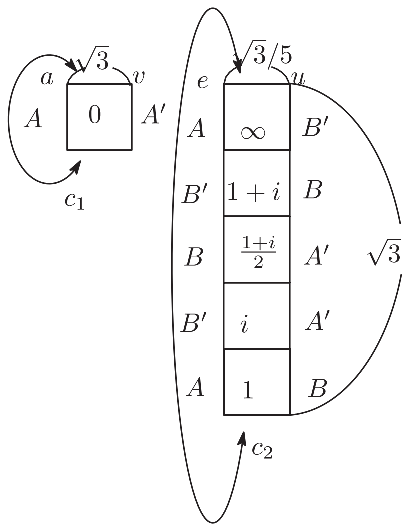

Fig. 5. Horospherical cross-sections of the cusps of $B^3 - S$.

To see the horoball neighborhoods of cusps of $S^2 \times I - T$, we will consider the tangle T_0 in $S^2 \times I$ as in Figure 3. Let P_2 be the right-angled ideal cuboctahedron and Y_1, Y_1', Y_2, Y_2', Y_3 and Y_3' the ideal squares of P_2 as shown in Figure 6. Put

$$f = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},\ g = \begin{pmatrix} -1\sqrt{2} & 1-2i\sqrt{2} \\ -2 & 3i-\sqrt{2} \end{pmatrix},\ h = \begin{pmatrix} 2i\sqrt{2} & -3-i\sqrt{2} \\ -3\sqrt{2} & 3i\sqrt{2} \end{pmatrix}.$$

The face Y_2 is identified to Y_1 by f, Y_3 is identified to Y_1' by g and Y_2' is identified to Y_3' by h. Put c_1, c_1', c_2 and c_2' be the cusps of $S^2 \times I - T_0$ as in Figure 3. In [1] Section 2, the following proposition is proved.

Proposition 4. *There is a homeomorphism from $S^2 \times I - T_0$ to $P_2/\{f^{\pm 1}, g^{\pm 1}, h^{\pm 1}\}$, which is a hyperbolic manifold with totally geodesic boundary. The ideal point* 0 (*resp.* $1 - i\sqrt{2}/2$) *corresponds to the cusp c_1 (resp. c_1'). The five ideal points ∞, $1/2 - i/\sqrt{2}$, $-i\sqrt{2}$, $(2 - i\sqrt{2})/3$ and 1 correspond to the cusp c_2. The five ideal points $(1 - 2i\sqrt{2})/3$, $1 - i\sqrt{2}$, $(2 - 2i\sqrt{2})/3$, $-i/\sqrt{2}$ and $(1 - i\sqrt{2})/3$ correspond to the cusp c_2'.*

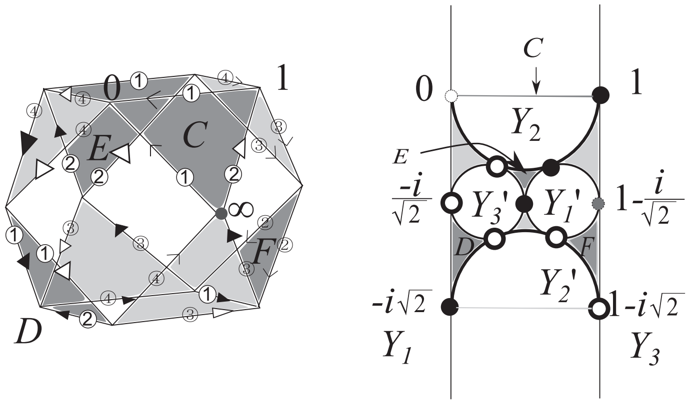

Fig. 6. The ideal polyhedral decomposition of $S^2 \times I - T_0$.

Proposition 5. *Let $N(c_j)$ be the horoball neighborhood of c_j such that the meridian length of $\partial N(c_j)$ is $\sqrt{3}$ ($j = 1, 2$). Then $N(c_j) \cap N(c_k) = \emptyset$ and*

$$N(c_j) \cap N(c_k') = \begin{cases} \{\textit{one point}\} & (j = k = 1) \\ \emptyset & (\textit{otherwise.}) \end{cases}$$

(Proof of Proposition 5.) Let $\mathcal{V}_2$ be the set of ideal vertices of P_2. We consider a set of horoballs $\{h_z^1 | z \in \mathcal{V}_2\}$ which is invariant under the action of the symmetry group of P_2,

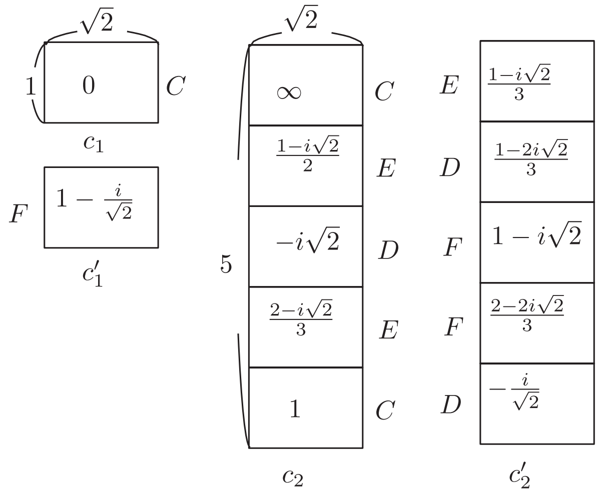

Fig. 7. Horospherical cross-sections of $S^2 \times I - T_0$.

such that h_z^1 is centered at z for each $z \in \mathcal{V}_2$ and $D(h_\infty^1) = 1$. Put $\alpha = \begin{pmatrix} \sqrt{2} & i \\ i & 0 \end{pmatrix}, \beta = \begin{pmatrix} -i & i \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}$. α is the $\pi/2$ rotation which fixes an ideal quadrilateral Y_1 and Y_1' of P_2. β is the $\pi/2$ rotation which fixes ideal quadrilaterals Y_2 and Y_2' in P_2. As $\alpha(h_\infty^1) = h^1_{-i\sqrt{2}}$, $D(h^1_{-i\sqrt{2}}) = 1$ by Lemma 1. By the symmetry of P_2, $D(h_0^1) = D(h^1_{-i\sqrt{2}}) = D(h^1_{1-i\sqrt{2}}) = 1$. As $\beta(h_\infty^1) = h^1_{-i/\sqrt{2}}$, $D(h^1_{-i/\sqrt{2}}) = 1/2$ by Lemma 1. By the symmetry of P_2, $D(h^1_{1-i\sqrt{2}/2})$ is also $1/2$. As $\beta^2 = \begin{pmatrix} -1\sqrt{2} & -i\sqrt{2} \\ -2 & 1-i\sqrt{2} \end{pmatrix}$, we have $\beta^2(h_\infty^1) = h^1_{(1-i\sqrt{2})/2}$ and $D(h^1_{(1-i\sqrt{2})/2}) = 1/4$ by Lemma 1. As $\beta\alpha^{-1} = \begin{pmatrix} 1 & 1-i\sqrt{2} \\ 1\sqrt{2} & -2 \end{pmatrix}$, we have $\beta\alpha^{-1}(h_\infty^1) = h^1_{(1-i\sqrt{2})/3}$ and $D(h^1_{(1-i\sqrt{2})/3}) = 1/3$ by Lemma 1. By the symmetry of P_2, $D(h^1_{(1-2i\sqrt{2})/3}) = D(h^1_{(2-2i\sqrt{2})/3}) = D(h^1_{(2-i\sqrt{2})/3}) = 1/3$. Glue $\{P_2 \cap \partial h_z^1 | z \in \mathcal{V}_2\}$ by f, g, h. We get horospherical cross-sections of the cusps as in Figure 7. The meridian lengths of horospherical cross-sections of the cusps c_1 and c_1' (resp. c_2 and c_2') are 1 (resp. 5).

Expand (resp. shrink) the horoball neighborhoods of c_1 and c_1' (resp. c_2 and c_2') until the meridian lengths of them are $\sqrt{3}$. Put $\{h_z | z \in \mathcal{V}_2\}$ be a set of horoballs such that h_z projects to $N(c_j)$ or $N(c_j')$. To check $N(c_1) \cap N(c_1') = \{\text{one point}\}$, $N(c_1) \cap N(c_2) = N(c_1) \cap N(c_2') = \emptyset$ and $N(c_2) \cap N(c_2') = \emptyset$, we consider $\alpha(P_2)$. By the definition of h_z^1, $\alpha(h_z^1) = h^1_{\alpha(z)}$. $\alpha(0) = \infty$ (resp. $\alpha(1 - i/\sqrt{2}) = 2/3 - 2i\sqrt{2}/3$) corresponds to c_1 (resp. c_1'). Thus $D(h_\infty) = D(h_\infty^1)/\sqrt{3}$ and $D(h_{(2-2i\sqrt{2})/3}) = \sqrt{3} D(h^1_{(2-2i\sqrt{2})/3})$. As $D(h_\infty^1) = 1$ and $D(h^1_{(2-2i\sqrt{2})/3}) = 1/3$, $D(h_\infty) = 1/\sqrt{3}$ and $D(h_{(2-2i\sqrt{2})/3}) = 1/\sqrt{3}$. Therefore $N(c_1) \cap N(c_1') = \{\text{one point}\}$ (see the left side of Figure 8).

Because $D(h_z) = \sqrt{3} D(h_z^1)/5$ for $z \in V_2 \setminus \{\infty,\ (2-2i\sqrt{2})/3\}$, we get

$$D(h_1) = D(h_0) = D(h_{-i\sqrt{2}}) = D(h_{1-i\sqrt{2}}) = \sqrt{3}/5,$$
$$D(h_{(1-i\sqrt{2})/3}) = D(h_{(1-2i\sqrt{2})/3}) = D(h_{(2-i\sqrt{2})/3}) = \sqrt{3}/15,$$
$$D(h_{-i/\sqrt{2}}) = D(h_{1-i/\sqrt{2}}) = \sqrt{3}/10,$$
$$D(h_{(1-i\sqrt{2})/2}) = \sqrt{3}/20.$$

Thus $N(c_1) \cap N(c_2) = N(c_1) \cap N(c_2') = \emptyset$ and $N(c_2) \cap N(c_2') = \emptyset$ (see the left side of Figure 8).

We consider $\beta(P_2)$. As $\beta(1 - i/\sqrt{2}) = \infty$ (resp. $\beta(0) = (1 - i\sqrt{2})/3$), ∞ (resp. $(1 - i\sqrt{2})/3$) corresponds to c_1' (resp. c_1). By the same way, we can get $N(c_1') \cap N(c_2) = N(c_1') \cap N(c_2') = \emptyset$ (see the right side of Figure 8).

5. Proof of main Theorem.

$S^2 \times I - T_0$ has two boundary components that we will call $\partial_{(}S^2 \times I - T_0)$ and $\partial_-(S^2 \times I - T_0)$, with the latter triangulated by the letter-labeled faces of Figure 6. By identifying the edges of these letter-labeled ideal triangles by f, g and h, we can get a four-punctured sphere $\partial_-(S^2 \times I - T_0)$ as in Figure 9. $S^2 \times I - T$ can be formed by gluing together $S^2 \times I - T_0$ and its mirror image $\overline{S^2 \times I - T_0}$ along $\partial_{(}S^2 \times I - T_0)$. The corresponding mirror image of $\partial_-(S^2 \times I - T_0)$ is triangulated by corresponding ideal triangles $\bar{C}$, $\bar{D}$, $\bar{E}$ and $\bar{F}$. The

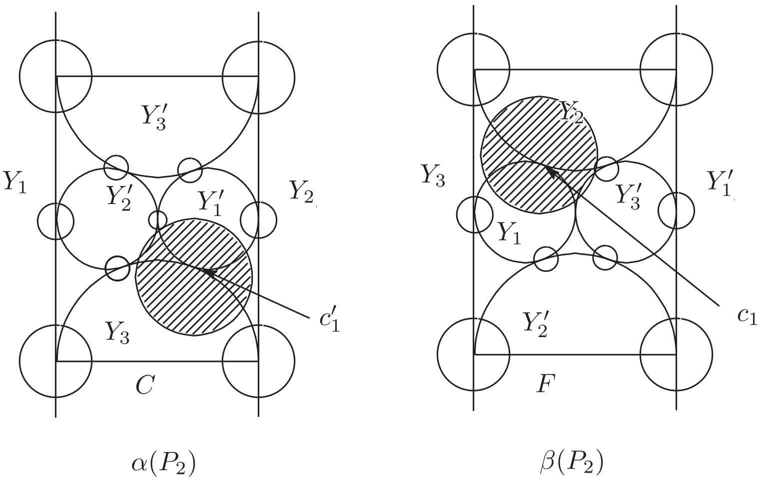

Fig. 8. Horoballs in P_2.

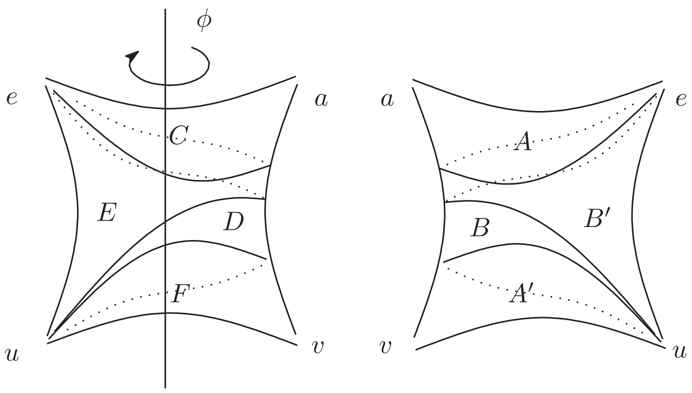

Fig. 9. Boundaries of the tangles.

horospherical cross-sections of the cusps of $S^2 \times I - T$ are as in Figure 10.

The colored ideal triangles A, A', B, B' in Figure 4 correspond to the totally geodesic boundary of $B^3 - S$. We identify the edges of these triangles by s and t. The resulting surface is a four-punctured sphere.

Take a base point for $\pi_1(B^3 - S)$ (resp. $\pi_1(S^2 \times I - T_0)$) on $\partial(B^3 - S)$ (resp. $\partial(S^2 \times I - T_0)$) high above the projection plane, and let its Wirtinger generators correspond in the usual way to labeled arcs of the diagram as in Figure 3. The ideal points in four-punctured sphere correspond to a, e, v, u as shown in Figure 9 ([1] Lemma 2.4, Proposition 2.7, 2.8).

We construct $S^3 - L$. Let T_i ($i = 1, \ldots, n$) be a copy of T and $\overline{S}$ a mirror image of S. Let C_i, D_i, E_i, F_i, $\overline{C_i}$, $\overline{D_i}$, $\overline{E_i}$, $\overline{F_i}$ be the faces on $\partial(S^2 \times I - T_i)$ corresponding to C, D, E, F, $\overline{C}$, $\overline{D}$, $\overline{E}$, $\overline{F}$. Glue $B^3 - S$ and $S^2 \times I - T_1$ by identifying A and C_1, B and D_1, B' and E_1, A' and F_1. Glue $S^2 \times I - T_i$ and $S^2 \times I - T_i$ ($i = 1, \ldots, n-1$) by identifying the faces $\overline{C_i}$ (resp. $\overline{D_i}$, $\overline{E_i}$, $\overline{F_i}$) and C_i (resp. D_i, E_i, F_i). Glue $B^3 - \overline{S}$ and $S^2 \times I - T_n$ by identifying $\overline{A}$ and $\overline{C_n}$, $\overline{B}$ and $\overline{D_n}$, $\overline{B'}$ and $\overline{E_n}$, $\overline{A'}$ and $\overline{F_n}$. By Figure 5 and 10, the horospherical cross-sections of the cusps of $S^3 - L$ are as in Figure 11.

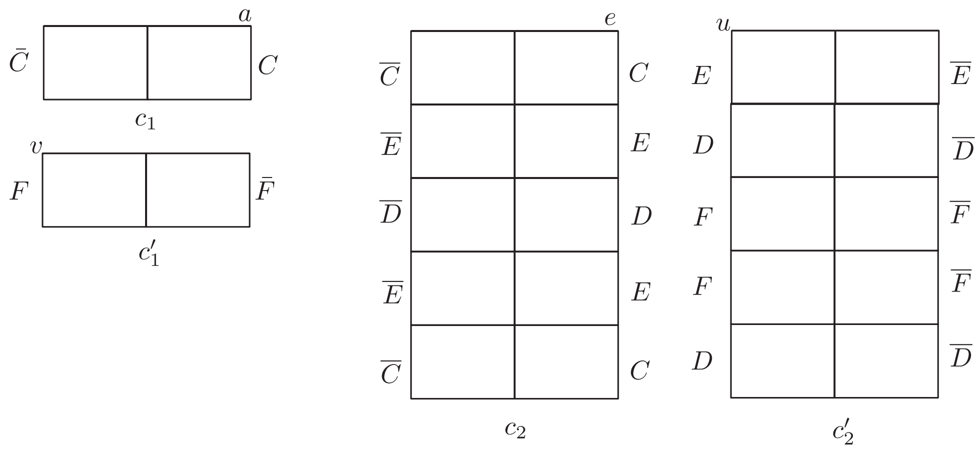

Fig. 10. The horospherical cross-section of $S^2 \times I - T$.

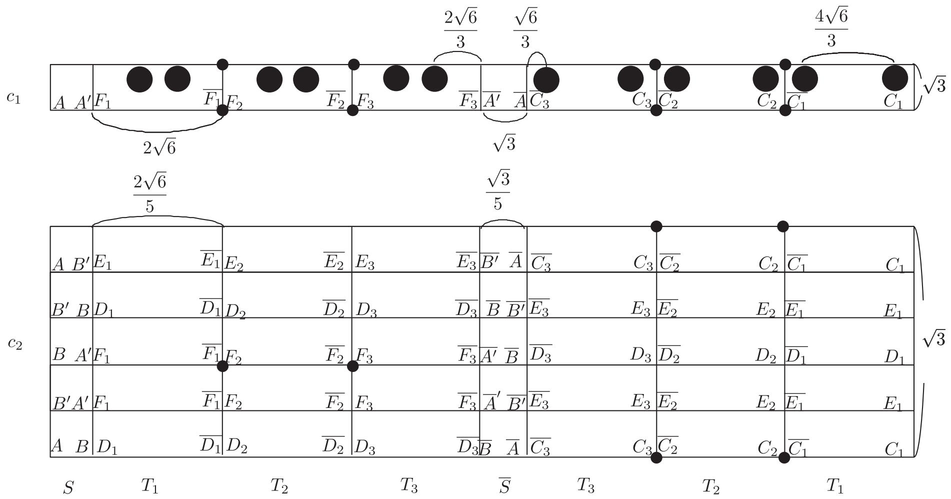

Fig. 11. The horospherical cross-sections of the cusps of $S^3 - L$ ($n = 3$).

To construct $S^3 - L_m$, cut along $S^{(m-1)}$ and $S^{(m)}$ and re-glue by using a symmetry ϕ on $\partial(S^2 \times I - T_m)$ as in Figure 9. Let $e_m(C)$ (resp. $e_m(\overline{C})$, $e(A)$, $e(\overline{A})$) be an edge of C_m (resp. $\overline{C_m}$, A, $\overline{A}$) which is incident to the punctures corresponding to e and a, and $e_m(F)$ (resp. $e_m(\overline{F})$, $e(A')$, $e(\overline{A'})$) an edge of F_m (resp. $\overline{F_m}$, A', $\overline{A'}$) which is incident to the punctures corresponding to u and v. The intersection of these edges and the horospherical cross-sections of $S^3 - L$ are the black points as in Figure 11. If $2 \leq m \leq n-1$, $e_m(C)$, $e_m(F)$, $e_m(\overline{C})$ and $e_m(\overline{F})$, are identified to $e_{m-1}(\overline{C})$, $e_{m-1}(\overline{F})$, $e_m(C)$ and $e_m(F)$ respectively. If $m = 1$, $e_1(C)$, $e_1(F)$, $e_1(\overline{C})$ and $e_1(\overline{F})$ are identified to $e(A)$, $e(A')$, $e_2(C)$ and $e_2(F)$. If $m = n$, $e_n(C)$, $e_n(F)$, $e_n(\overline{C})$ and $e_n(\overline{F})$ are identified to $e_{n-1}(\overline{C})$, $e_{n-1}(\overline{F})$, $e(\overline{A})$ and $e(\overline{A'})$. The horospherical cross-sections of the cusps of L_m are as in Figure 12.

If we expand the horoball neighborhoods of c_1 and c_2 in $S^3 - L$ until the meridian lengths of $\partial N(c_1)$ and $\partial N(c_2)$ are $\sqrt{3}$, $N(c_1)$ collides itself at $2n$ points and $N(c_1) \cap N(c_2) = N(c_2) \cap N(c_2) = \emptyset$. By performing mutation along $S^{(m-1)}$ and $S^{(m)}$, the cusps c_1 and c_2 are exchanged

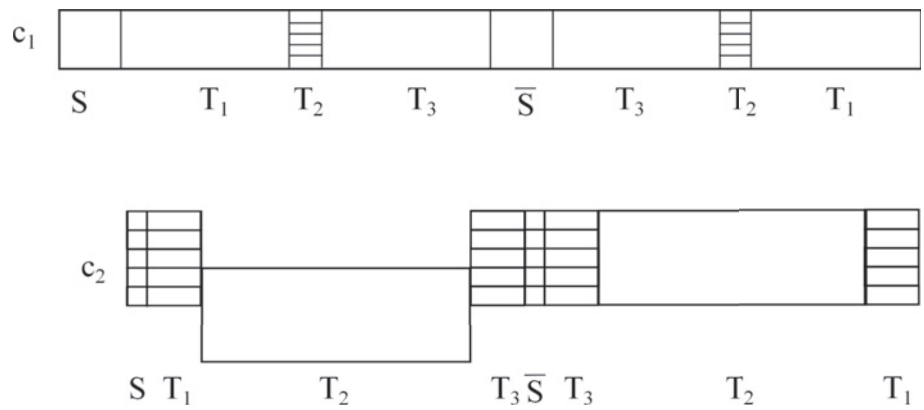

Fig. 12. The horospherical cross-sections of the cusps of $S^3 - L_2$ $(n = 3)$.

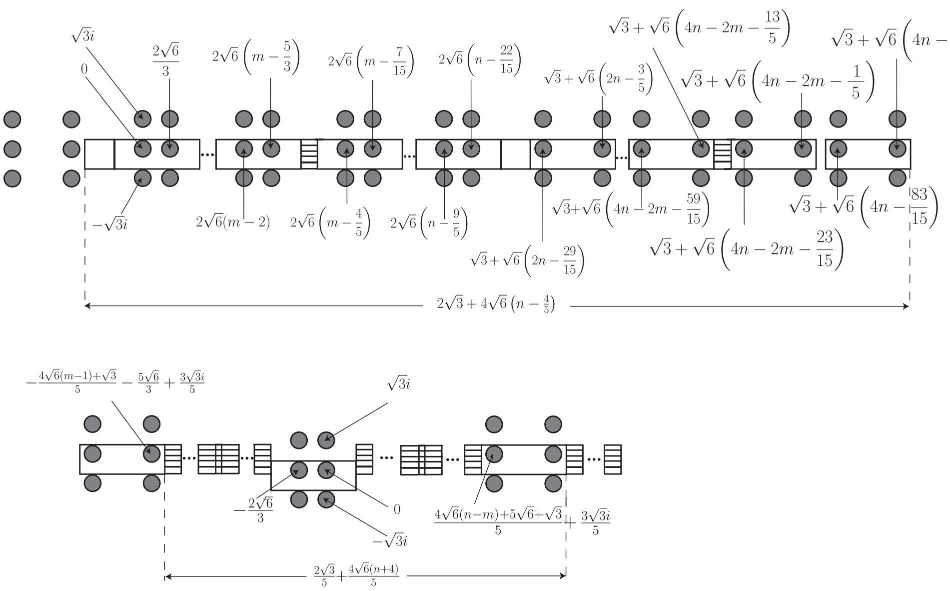

Fig. 13. $\mathcal{H}(S^3 - L_m)$.

in $S^2 \times I - T_m$. In $S^3 - L_m$, $N(c_1)$ collides itself at $2n-2$ points, $N(c_2)$ collides itself at 2 points, and $N(c_1) \cap N(c_2) = \emptyset$.

Let h_∞ be the horoball centered at ∞ with Euclidean height 1. Lift $N(c_1)$ and $N(c_2)$ to the upper half space model such that h_∞ is a lift of $N(c_1)$ (resp. $N(c_2)$). The horoballs which collide to h_∞ are as in the upper (resp. lower) side of Figure 13. Put

$$\begin{aligned} S_{m,1} = \Big\{ & 2\sqrt{6}(k-1),\ 2\sqrt{6}\left(k-\frac{2}{3}\right),\ 2\sqrt{6}\left(l-\frac{9}{5}\right),\ 2\sqrt{6}\left(l-\frac{22}{15}\right), \\ & \sqrt{3}\sqrt{6}(4n-2k-\frac{11}{5}),\ \sqrt{3}\sqrt{6}\left(4n-2k\frac{53}{15}\right),\ \sqrt{3}\sqrt{6}\left(4n-2l-\frac{3}{5}\right), \\ & \sqrt{3}\sqrt{6}\left(4n-2l-\frac{29}{15}\right) \mid\ k=1,\cdots,m-1,\ l=m,\cdots,n \Big\} \end{aligned}$$

$$(\text{resp. } S_{m,2} = \Big\{0,\ -\frac{2\sqrt{6}}{3},\ \frac{4\sqrt{6}(n-m)\sqrt{6}\sqrt{3}}{5}\frac{3\sqrt{3}i}{5}, \\ -\frac{4\sqrt{6}(m-1)\sqrt{3}}{5} - \frac{5\sqrt{6}}{3}\frac{3\sqrt{3}i}{5}\Big\})$$

and

$$X_{m,1} = \left\{x\sqrt{3}ai\left(2\sqrt{3}\sqrt{6}\left(n-\frac{4}{5}\right)\right)b \mid\ x \in S_{m,1}, a, b, \in \mathbb{Z}\right\}$$

$$(\text{resp. } X_{m,2} = \left\{x\sqrt{3}ai\left(\frac{2\sqrt{3}}{5}\frac{4\sqrt{6}(n)}{5}\right)b \mid\ x \in S_{m,2}, a, b, \in \mathbb{Z}\right\}).$$

The set of centers of horoballs which collide to h_∞ is $X_{m,1}$ (resp. $X_{m,2}$). We remark that there is no pair of points of $S_{m,k}$ $(k = 1, 2)$ whose distance is $\sqrt{3}$.

Suppose that there exists $f \in PSL(2,\mathbb{C})$ such that $f(\mathcal{H}(S^3 - L_m)) = \mathcal{H}(S^3 - L_{m'})$ $(m \neq m')$. In the universal cover of $S^3 - L_m$, we may assume that ∞ corresponds to the cusp c_2 of $S^3 - L_m$ and that $f(\infty) = \infty$.

Case 1. Suppose that $f(\infty)$ corresponds to the cusp c_1 of $S^3 - L_{m'}$. Then $f(X_{m,2}) = X_{m',1}$. We consider the ideal points 0, $\sqrt{3}i$, $\frac{4\sqrt{6}(n-m)\sqrt{6}\sqrt{3}}{5}\frac{3\sqrt{3}i}{5} \in X_{m,2}$. Put $z = f(0) \in X_{m',1}$. As $\left|f(0) - f(\sqrt{3}i)\right| = \sqrt{3}$, we get $f(\sqrt{3}i) = z\sqrt{3}i$ or $z - \sqrt{3}i$. Thus we have $f\left(\frac{4\sqrt{6}(n-m)\sqrt{6}\sqrt{3}}{5}\frac{3\sqrt{3}i}{5}\right) = z\frac{4\sqrt{6}(n-m)\sqrt{6}\sqrt{3}}{5}\frac{3\sqrt{3}i}{5}$ or $z\frac{4\sqrt{6}(n-m)\sqrt{6}\sqrt{3}}{5} - \frac{3\sqrt{3}i}{5}$. Because $f(0) = z \in X_{m',2}$, $z\frac{4\sqrt{6}(n-m)\sqrt{6}\sqrt{3}}{5} \pm \frac{3\sqrt{3}i}{5} \notin X_{m',1}$. This case cannot occur.

Case 2. Suppose that $f(\infty)$ corresponds to the cusp c_2 of $S^3 - L_{m'}$. Then $f(X_{m,2}) = X_{m',2}$. Consider the points 0, $\sqrt{3}i$, $-\frac{2\sqrt{6}}{3}$, $\frac{4\sqrt{6}(n-m)\sqrt{6}\sqrt{3}}{5}\frac{3\sqrt{3}i}{5} \in X_{m,2}$. We may assume $f(0) \in S_{m',2}$. As $\left|f(0) - f(\sqrt{3}i)\right| = \sqrt{3}$, $\left|f(0) - f(-\frac{2\sqrt{6}}{3}i)\right| = \frac{2\sqrt{6}}{3}$ and $f(0)$, $f(\sqrt{3}i)$, $f(-\frac{2\sqrt{6}}{3}) \in X_{m',2}$, there are two cases:

Case 2.1. $f(0) = 0$, $f(\sqrt{3}i) = \sqrt{3}i$ and $f(-\frac{2\sqrt{6}}{3}) = -\frac{2\sqrt{6}}{3}$.

Case 2.2. $f(0) = -\frac{2\sqrt{6}}{3}$, $f(\sqrt{3}i) = -\frac{2\sqrt{6}}{3} - \sqrt{3}i$ and $f(-\frac{2\sqrt{6}}{3}) = 0$.

In Case 2.1, we have $f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. As $m \neq m'$, $f(\frac{4\sqrt{6}(n-m)\sqrt{6}\sqrt{3}}{5}\frac{3\sqrt{3}i}{5}) = \frac{4\sqrt{6}(n-m)\sqrt{6}\sqrt{3}}{5}\frac{3\sqrt{3}i}{5} \notin X_{m',2}$. This case cannot occur.

In Case 2.2, we have $f = \begin{pmatrix} -i & -\frac{2\sqrt{6}i}{3} \\ 0 & i \end{pmatrix}$. $f(\frac{4\sqrt{6}(n-m)\sqrt{6}\sqrt{3}}{5}\frac{3\sqrt{3}i}{5}) = -\frac{4\sqrt{6}(n-m)\sqrt{6}\sqrt{3}}{5} - \frac{3\sqrt{3}i}{5} - \frac{2\sqrt{6}}{3} \notin X_{m',2}$. This case cannot occur.

Therefore if $m \neq m'$, $\mathcal{H}(S^3 - L_m) \neq \mathcal{H}(S^3 - L_{m'})$ up to $PSL(2,\mathbb{C})$. By Proposition 1, if $m \neq m'$, $S^3 - L_m$ and $S^3 - L_{m'}$ are incommensurable. This completes the proof of Theorem 1.

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